THE PROBLEM OF SYNCHRONIZATION OF DYNAMICAL SYSTEMS

(PROBLEMA SINKHRONIZATSII DINAMICHESKIKH SISTEM)

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Synchronization is a phenomenon which is observed in man-made as well as in natural objects — electric generators, vacuum-tube oscillators, mechanical vibrators, pendulum clocks, musical instruments and certain biological systems. We give here a general formulation of the problem of synchronization of dynamic objects and investigate their peculiarities; we also enumerate the principal specific problems and applications. A mathematical apparatus is indicated which is useful for the study of a basic class of problems of synchronization — the problems of coordinated functioning of some almost identical self-oscillating objects, weakly coupled to one another. The observed tendency of such objects, of the most diverse sort, toward synchronous motion finds it mathematical expression in the fact that the governing system of differential equations with periodic coefficients, as a rule, allows of a stable periodic solution. We give a short review of the work on the theory of synchronization of dynamic systems and we then enumerate the problems that have as yet not been solved.

The phenomenon of synchronization may be described as follows: a number of man-made or natural objects, which in the absence of intercoupling oscillate or rotate with various frequencies (angular velocities), begin to move with identical or multiple frequencies (angular velocities) upon the application of at times very weak intercoupling. In the process, a definite phase relationship is established between the respective oscillations and rotations.

Particular cases of the phenomenon of synchronization have been known for a long time. Ch. Huygens at the beginning of the second half of the seventeenth century established that a pair of pendulum clocks, beating differently, would synchronize themselves when they were attached to a thin beam instead of to the wall [1].

Rayleigh observed synchronization in acoustical and electro-acoustical systems at the end of the nineteenth century. Observing two organ pipes with holes distributed in a row, he found that for sufficiently small mistuning the pipes would sound in unison, i.e. there would occur a mutual synchronization of the two self-oscillating systems. Sometimes the pipes caused complete silencing of one another. An analogous phenomenon was also observed by Rayleigh for two tuning forks with electro-magnetic excitation. The forks were coupled together either electrically, mechanically by means of an elastic wire, or, finally by means of a box resonator [2].

Later, approximately at the beginning of the present century, synchronization phenomena were discovered in electric networks and in certain electromechanical systems. Until recent times, the principal technological applications of synchronization (synchronization of electrical generators and vacuum-tube oscillators) were associated with these items.

In 1947 – 1948 the phenomenon of the self-synchronization of mecanically unbalanced vibrators, fixed to a single vibrating element, was discovered in the USSR [3 and 4]. It turned out that such vibrators, represented in the simplest case by unbalanced rotors driven by any sort of motor of asynchronous type, would, under certain circumstances, operate synchronously, in spite of the possible difference of vibrator parameters and in spite of the absence of any kinematic or electric coupling between the rotors.

At present time, self-synchronization, and likewise the related phenomenon of the sustaining of the rotation of unbalanced rotors by vibrational means [5 and 6], finds a very wide application in the new construction of vibrating machinery within the USSR, as well as beyond its borders [7, 8 and 9].

The effect of vibrational excitation and sustaining of rotation is, in essence, also used in nuclear technology in the design of cyclic accelerators of charged particles [9]. In the Soviet Union a number of means of forced electrical synchronization and phasing of rotations of vibrators have been suggested [10 and]].

Finally N. Wiener has supposed that the phenomenon of synchronization lies at the basis of the excitation of alpha-rhythm of the brain, and likewise he gives a far-reaching suggestion of the role of this phenomenon in the processes of self-organization and self-reproduction of certain biological objects, in particular, in the processes of the evolution of malignant tumors [12].

The technological problem of synchronization is a particular case of a more general problem, the guaranteeing of the concordant functioning of a number of objects. In this connection, in certain cases synchronization and specific phasing takes place by virtue of the natural coupling (in the wide sense of the word) which is already present in the system. Thus, for example, in the problem of the synchronization of generators of electrical or mechanical oscillations, synchronization is frequently realized because of the properties of the system itself – the generator and load. This type of synchronization is usually called self-synchronization. In other cases, the effect of synchronization and phasing is obtained by means of the introduction of auxiliary synchronizing elements (forced synchronization).

The most important examples of problems of synchronization are:

1. To obtain the conditions of synchronization and proper phasing of mechanical vibrators. This is one of the main problems that arises in the design of new types of modern vibrating machinery, as sifters, conveyers, crushers, millers, etc.

2. To investigate the conditions of stability in the parallel operations of a number of electrical generators on a common load. This problem is of particular significance in connection with the integration of complicated electrical-power systems.

3. To obtain the conditions of synchronization and definite phasing of self-oscillations excited in a number of vacuum-tube oscillators.

To the same class of problems belong the following: the investigation of the peculiarities of the motion of rotating elastic shafts with unbalanced disks, the dynamic analysis of special automatic balances for the compensation of unbalanced high-speed rotors, the study of the behavior of a number of unbalanced machines fixed to a common foundation or to common supporting structures connected between themselves, the investigation of the principles of operation of a number of acoustical instruments, in particular the peculiarities of the sound of certain musical instruments, and the investigation of certain biological phenomena.

The short review of the history and technological use of synchronization that has been introduced shows that everywhere where there are oscillatory processes the problem of synchronization arises sooner or later.

We present below an attempt to examine this problem in a certain general formulation. In other words, we attempt to study the general properties of the behavior of interconnected various nature objects of the same type.

1. Certain problems in the synchronization of dynamic objects.

1. Synchronization of mechanical vibrators. The problem of the synchronization of mechanical vibrators is one of the main problems in the theory of vibrating machinery. We formulate this problem for the simplest possible case, the self-synchronization of unbalanced vibrators fixed to an absolutely rigid platform having one degree of freedom (Fig. 1).

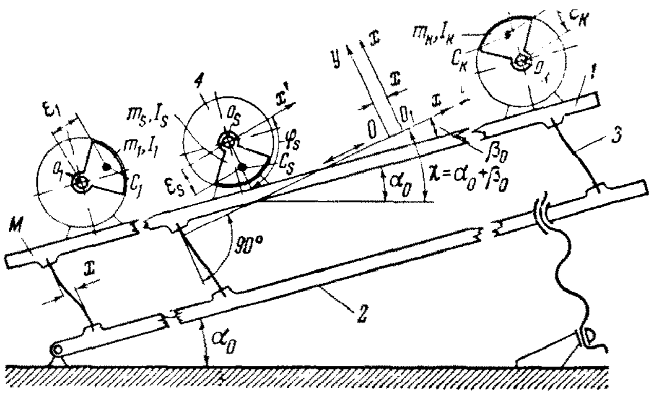

Fig. 1

The vibrating element of the machine 1 (rigid platform) is connected to an immovable foundation 2 by a system of planar elastic supports 3 (springs). The axes of the springs are assumed to be inextensible and therefore the platform can translate only in a direction perpendicular to these axes. On the platform are fixed a certain number k of unbalanced vibrators 4 in the form of unbalanced rotors, whose axes are perpendicular to the plane of oscillation of the platform and which are rotationally driven by some sort of motors of asynchronous type. The state of the system is characterized by the deflection of the platform x from a position of static equilibrium and by the angle of rotation of the rotors of the vibrator φ_s, measured clockwise.

The differential equations of motion of the system have the form [3]

$$I_s\varphi_s^{\cdot\cdot} = m_s\varepsilon_s[x^{\cdot\cdot}\sin\varphi_s + g\cos(\varphi_s - \chi)] + L_s(\varphi_s^{\cdot}) - R_s(\varphi_s^{\cdot})$$

$$Mx^{\cdot\cdot} + k_x x^{\cdot} + c_x x = -\sum_{s=1}^{k} m_s\varepsilon_s(\cos\varphi_s)^{\cdot\cdot} \qquad (s = 1, \ldots, k) \qquad (1.1)$$

Here $L_s(\varphi_s^{\cdot})$ is the rotational moment of the motor; $R_s(\varphi_s^{\cdot})$ is the moment of the forces resisting rotation of the rotor of the vibrator; m_s, ε_s and I_s are, respectively, the mass, eccentricity, and moment of inertia around the axis of rotation of the rotor of the sth vibrator; M is the mass of the system, k_x is the coefficient of viscous resistance, c_x is the rigidity of the elastic system, g is the acceleration of gravity, χ is the angle between the direction of the x-axis and the horizontal.

The problem consists of establishing the conditions under which all the vibrators will rotate with the same mean absolute angular velocity, in spite of the absence of any direct connections between their rotors, the different parameters characterizing the vibrators and the forces acting on them. In other words, what is involved is a clarification of the conditions of existence and stability, and, likewise, the finding of the (even approximate) solutions of the system (1.1) of the form

$$\varphi_s = \sigma_s\,[\omega t + \psi_s(\omega t)] \quad (s = 1, \ldots, k), \qquad x = x(\omega t) \tag{1.2}$$

Here ω is the absolute value of the mean rotational velocity of the vibrators, $\psi_s(\omega t)$ and $x(\omega t)$ are periodic functions of time t with period $2\pi/\omega$, and each of the quantities σ_s is either equal to 1 or −1; the first case corresponding to a rotation of the sth vibrator in a positive direction and the second case to a rotation in the negative direction.

A motion of the type (1.2) we shall call synchronous. Sometimes our interest will also be in the investigation of multiple-synchronous motion when the mean value $|\varphi_s^{\cdot}|$ is equal to $n_s\,\omega$, n_s being a positive integer.

The modulus of the angular velocity of the synchronous rotation ω is not known beforehand and is subject to determination in the process of solving the problem.

It is not hard to see that if in Equations (1.1) one goes over from the dependent variables φ_s to new variables ψ_s in accordance with Formulas (1.2) and to a "nondimensional time" $\tau = \omega t$, then the problem reduces to the establishment of conditions of existence and stability of periodic (with period 2π) solutions of a system of $k + 1$ nonlinear equations of the second order with periodic coefficients. In its general form this problem is extremely complicated. However, under specific assumptions [3] one may introduce a small parameter into the system and by this means essentially simplify the investigation, the transformed system being represented in the form

$$\begin{aligned} I_s\psi_s'' + \frac{k_s}{\omega}\psi_s' &= \mu\sigma_s\Psi_s(\psi_s, x'', \tau) \qquad (s = 1, \ldots, k) \\ x'' + \frac{c_x}{M\omega^2}x &= -\sum_{s=1}^{k}\frac{m_s\varepsilon_s}{M}\,[\cos(\tau + \psi_s)]'' - \mu\frac{k_x^*}{M\omega}x' \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} \mu\sigma_s\Psi_s(\psi_s, x'', \tau) = m_s\varepsilon_s\left[x''\sin(\tau + \psi_s) + \frac{g\sigma_s}{\omega^2}\cos(\tau + \psi_s - \sigma_s\chi)\right] + \\ + \frac{\sigma_s}{\omega^2}\,[L_s(\sigma_s\omega) - R_s(\sigma_s\omega)] \end{aligned} \tag{1.4}$$

$$k_x = \mu k_x^*, \qquad k_s = k_s^* + k_s^\circ, \qquad k_s^* = -\left(\frac{dL_s}{d\varphi_s^{\cdot}}\right)_{\varphi_s^{\cdot} = \sigma_s\omega}, \qquad k_s^\circ = \left(\frac{dR_s}{d\varphi_s^{\cdot}}\right)_{\varphi_s^{\cdot} = \sigma_s\omega}$$

μ is a small parameter and where, as below, we denote differentiation with respect to $\tau = \omega t$ by means of a prime. In addition, in obtaining Equations (1.3) we have taken

$$\begin{aligned} L_s[\sigma_s\omega(1 + \psi_s')] &\approx L_s(\sigma_s\omega) - k_s^*\sigma_s\omega\psi_s' \\ R_s[\sigma_s\omega(1 + \psi_s')] &\approx R_s(\sigma_s\omega) + k_s^\circ\sigma_s\omega\psi_s' \end{aligned} \tag{1.5}$$

which corresponds to an assumption on the nearness of the motion of the vibrators to uniform rotation, i.e. on the smallness of ψ_s' as compared to unity. Usually $k_s^* > 0$ and $k_s^\circ > 0$.

An important property of the system (1.3) is the fact that for $\mu = 0$ the first k equations (the equations of motion of the synchronizable objects) turn out to be independent of one another, and likewise of the latter equation of (1.3). A second property is the presence of a k-fold root $\rho = 1$ in the characteristic equation of the variational system corresponding to solutions of the generating system. In this circumstance it is not difficult to verify that simple elementary divisors correspond to the multiple roots. These two properties are inherent in many problems of the synchronization of dynamic systems (see below).

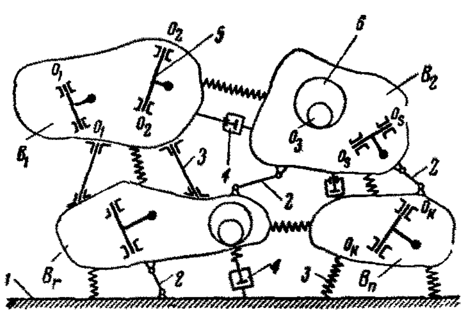

Fig. 2

The problem that has been posed may be generalized in an essential way, so for example, the vibrating machine (Fig. 2) may be not one but a number of rigid bodies B_r connected together by an immovable foundation 1, by certain geometric couplings 2, and likewise by elastic 3 and damping 4 elements. Among the vibrators there may be not only the simple unbalanced vibrators 5 described above, but also "planetary vibrators" 6, and the axes of the vibrators may be arbitrarily oriented in space. Sometimes it is of interest to investigate cases where the aforementioned rigid bodies may collide during the motion. However, at all times the matter at hand is the clarification of the conditions of existence and stability of motions of the type (2.2), that is, of synchronous motions.

2. The dynamics of an automatic balance for the equilibrating of rotating rotors. One of the possible forms of the balance [13 to 15] is shown schematically in Fig.3a. Onto a flexible rotating shaft 1 a disk 2 is attached, its center of gravity C does not lie on the axis of the shaft AOB. The disk contains a cylindrical or toroidal cavity which is filled with oil and whose axis coincides with the tangent to the axis of the shaft at the point of support O_1. A few balls 3 are placed in the cavity. Under specific conditions they arrange themselves in the rotating disk in such a way that they compensate the unbalance of the disk and thereby eliminate oscillations of the shaft and the transmission of dynamic loads to its bearings.

The equations of motion of the system have the form

$$R\ddot{\varphi}_s + \beta_0 R(\dot{\varphi}_s - \omega) = \ddot{x}\sin\varphi_s + \ddot{y}\cos\varphi_s \qquad (s = 1, \ldots, k)$$

$$M\ddot{x} + \beta\dot{x} + cx = M^\circ r\omega^2\cos\omega t - mR\sum_{s=1}^{k}(\cos\varphi_s)^{\cdot\cdot} \tag{1.6}$$

$$My^{\cdot\cdot} + \beta y^{\cdot} + cy = -M^{\circ} r \omega^2 \sin \omega t + mR \sum_{s=1}^{k} (\sin \varphi_s)^{\cdot\cdot} \qquad \begin{matrix}(1.6)\\ \text{contin.}\end{matrix}$$

$$(M = km + M^{\circ})$$

(The above equations were introduced in [15] in other notation).

Here (see Fig. 3b) x and y are the coordinates of the center of the disk O_1 in a fixed system of axes xOy, whose origin is at the point of intersection of the plane of the disk with the axis of the bearings; φ_s is the angle between the straight line joining the center of the disk and the center of the s ball and the direction of the x-axis, measured clockwise; M°, m and M are the masses of the disk, the ball and the entire system, respectively; r is the eccentricity of the disk, R is the distance from the center of the balls to the axis of the shaft; ω is the angular velocity of rotation of the shaft; β_0 and β are the coefficients of viscous resistance; c is the stiffness of the shaft in bending with respect to a force applied at the point O_1.

The problem reduces to the clarification of the conditions of existence and stability of the solutions of the system (1.6) of the form

$$\varphi_s = \omega t + \psi_s(\omega t) \qquad (s = 1, \ldots, k),$$

$$x = x(\omega t), \qquad y = y(\omega t) \quad (1.7)$$

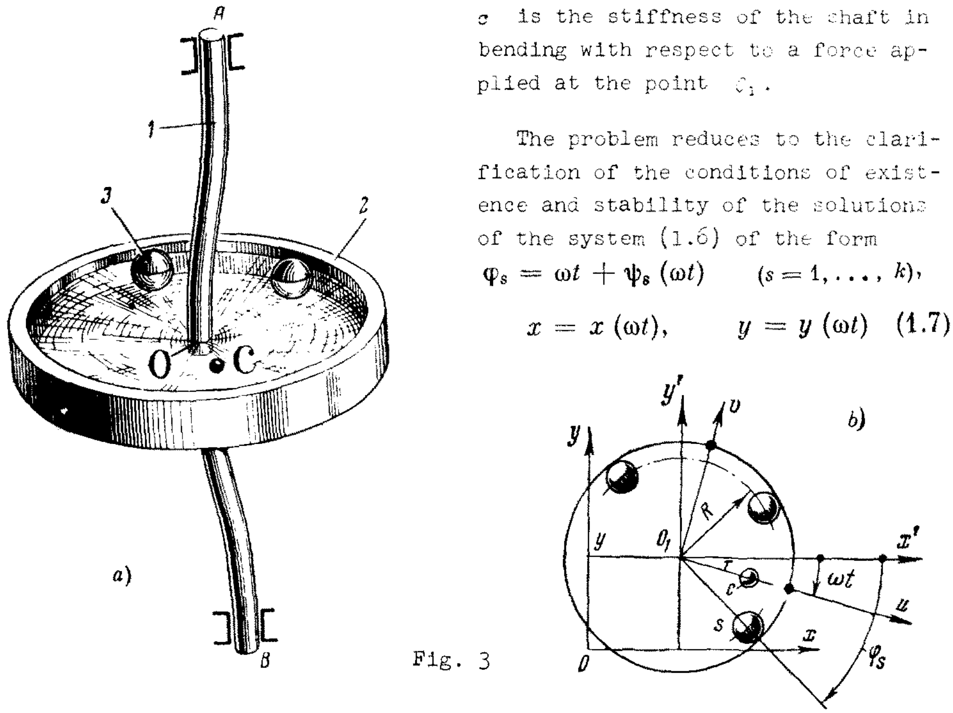

Fig. 3

Here ψ_s, x and y are periodic functions of the time t with period $2\pi/\omega$. In this case, of particular interest are those solutions of (1.7) in which $x(\omega t) \approx y(\omega t) \approx 0$, that is, solutions corresponding to the self-equilibration of the system, when oscillations of the shaft are absent.

Along with the great similarities in the formulation of the present and previous problems there are also differences, which are associated with the fact that the initial equations (1.1) for the problem of self-synchronization were autonomous, whereas the system (1.6) is nonautonomous. In the first case the frequency ω was assumed to be unknown beforehand, while in the second case it was taken as given. We note, however, that this difference may be eliminated if one assumes that the angle of rotation of the shaft φ is not given and if one attaches to the system (1.6) the equation of motion of the motor rotating the shaft.

If in Equations (1.6) one uses Formulas (1.7) to pass from the variables φ_s to new variables ψ_s and the nondimensional time $\tau = \omega t$, then this system may be represented in a form which is analogous to the systems (1.3) for the problem of the self-synchronization of vibrators

$$\psi_s'' + \frac{\beta_s}{\omega}\psi_s' = \mu\Phi(\psi_s, x'', y'', \tau) \qquad (s = 1, \ldots, k)$$

$$x'' + \frac{c}{M\omega^2}x = \frac{M^\circ}{M}r\cos\tau - \frac{m}{M}R\sum_{s=1}^{k}[\cos(\tau+\psi_s)]'' - \mu\frac{\beta^*}{M\omega}x'$$

$$y'' + \frac{c}{M\omega^2}y = -\frac{M^\circ}{M}r\sin\tau + \frac{m}{M}R\sum_{s=1}^{k}[\sin(\tau+\psi_s)]'' - \mu\frac{\beta^*}{M\omega}y' \qquad (1.8)$$

Here

$$\mu\Phi = \frac{1}{R}[x''\sin(\tau+\psi_s) + y''\cos(\tau+\psi_s)], \qquad \mu\beta^* = \beta \qquad (1.9)$$

whereby the quantity μ can, as before, be considered as a small parameter. This, clearly, corresponds to an assumption on the smallness of the deflection of the center of the disk x and y in comparison with the length of R, and likewise on the smallness of the coefficient $\beta/M\omega$ which characterizes the resistive force in the oscillation of the disk.

Hence the problem reduces to the clarification of the conditions of existence and stability of periodic (with period 2π) solutions of equations (1.8), particular interest being focused on the solutions $x \approx y \approx 0$.

3. Bending-torsional oscillations of a rotating shaft with unbalanced disks. We consider a system consisting of a multiply-supported shaft with an arbitrary number k of statically unbalanced disks (Fig. 4a). We shall assume that in its motion the shaft may perform not only bending but torsional oscillation; that is, we shall assume that the stiffness in torsion of the various portions of the shaft is finite. The shaft supports may be either rigid or flexible with nonidentical stiffnesses in various directions. Some of the disks may represent rotors of the motor which drive the shaft in rotation.

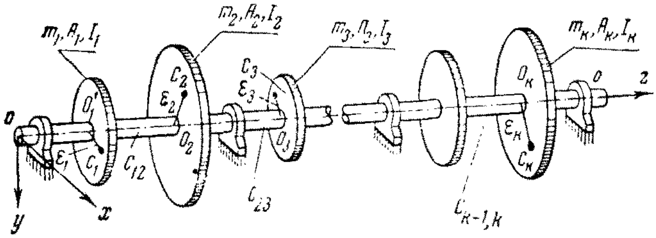

Fig. 4a

Let $Oxyz$ be a fixed system of rectangular coordinates, the z-axis of which is directed along the axis of the shaft bearings. Small oscillations of each disk are specified by cartesian coordinates x_s and y_s of the point of intersection of the planes of the disk with the axis of the shaft oo, and, likewise, by two Euler angles α_s and β_s, chosen in accordance Fig. 4b. The rotation of each disk is specified by the angle of its own rotation φ_s.

If gyroscopic terms are neglected and the forces of gravity are not taken into account, then the differential equations of motion of the system may be represented in the form

$$I_s\varphi_s^{\cdot\cdot} = m_s\varepsilon_s(x_s^{\cdot\cdot}\cos\varphi_s + y_s^{\cdot\cdot}\sin\varphi_s) - c_{s-1,s}(\varphi_s - \varphi_{s-1} - \varkappa_s + \varkappa_{s-1}) + c_{s,s+1}(\varphi_{s+1} - \varphi_s - \varkappa_{s+1} + \varkappa_s) + L_s - R_s \quad (s = 1, \ldots, k) \tag{1.10}$$

$$m_s x_s^{\cdot\cdot} + \sum_{j=1}^{k}(c_{sj}^{(x)}x_j + c_{sj}^{(x\alpha)}\alpha_j) = m_s\varepsilon_s(\varphi_s^{\cdot\cdot}\cos\varphi_s - \varphi_s^{\cdot 2}\sin\varphi_s) + Q_s^{(x)}$$

$$m_s y_s^{\cdot\cdot} + \sum_{j=1}^{k}(c_{sj}^{(y)}y_j + c_{sj}^{(y\beta)}\beta_j) = m_s\varepsilon_s(\varphi_s^{\cdot\cdot}\sin\varphi_s + \varphi_s^{\cdot 2}\cos\varphi_s) + Q_s^{(y)}$$

$$A_s\alpha_s^{\cdot\cdot} + \sum_{j=1}^{k}(c_{sj}^{(\alpha)}\alpha_j + c_{sj}^{(x\alpha)}x_j) = Q_s^{(\alpha)}$$

$$A_s\beta_s^{\cdot\cdot} + \sum_{j=1}^{k}(c_{sj}^{(\beta)}\beta_j + c_{sj}^{(y\beta)}y_j) = Q_s^{(\beta)}$$

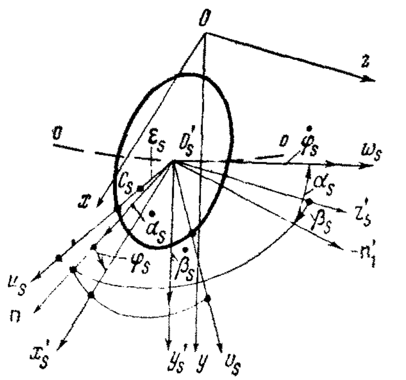

Fig. 4b

Here m_s, ε_s, A_s and I_s are, respectively, the mass, the eccentricity, and the equatorial and polar moments of inertia of the s disk; $c_{s,s+1}$ is the stiffness under torsion of a portion of the shaft between the s and and the $s+1$ disk, whereby $c_{01} = c_{k,k+1} = 0$;

$$c_{sj}^{(x)},\ c_{sj}^{(y)},\ c_{sj}^{(\alpha)},\ c_{sj}^{(\beta)},\ c_{sj}^{(x\alpha)},\ c_{sj}^{(x\beta)}$$

are respectively the stiffnesses of the shaft under bending with account taken of the pliability of the supports; $\varkappa_s$ is the value of the angles φ_s, under which the elastic twisting moments over the span of the shaft, are equal to zero (these angles, determined with accuracy up to the constant rotation $\varkappa_0$, characterize the directions of the eccentricity vectors $\varepsilon_s = O_s'C_s$ of the disks for the untwisted shaft); $Q_s^{(x)}$, $Q_s^{(y)}$, $Q_s^{(\alpha)}$ and $Q_s^{(\beta)}$ are forces and moments of internal and external resistance to oscillations of the shaft which may depend on all of the generalized coordinates and velocities of the system, the coordinates φ_s and the velocities $\varphi_s^{\cdot}$ entering into the expressions for Q only in the form of differences $\varphi_s - \varphi_j$ and $\varphi_s^{\cdot} - \varphi_j^{\cdot}$; L_s and R_s are, respectively, the rotational moments of the motors and the moments of forces resisting to the rotation (it is usually sufficient to assume that these moments are functions of $\varphi_s - \varphi_j$, $\varphi_s^{\cdot} - \varphi_j^{\cdot}$ and $\varphi_s^{\cdot}$, and likewise, are possibly periodic functions φ_s with period of 2π).

The problem consists of establishing conditions for existence of stability, and likewise of calculating with some degree of accuracy the synchronous motion of the system, that is motions of the form

$$\varphi_s = \omega t + \psi_s(\omega t), \quad x_s = x_s(\omega t), \quad y_s = y_s(\omega t)$$
$$\alpha_s = \alpha_s(\omega t), \quad \beta_s = \beta_s(\omega t) \tag{1.11}$$

Here ψ_s, x_s, ϑ_s, α_s and β_s are periodic functions of time with period $2\pi/\omega$. In this case, as in the problem of the synchronization of vibrators, the value of the synchronous frequency ω, generally speaking, is not known beforehand and must be determined in the course of the solution of the problem. As previously, a transformation to the variables ψ_s and a passage to nondimensional time $\tau = \omega t$ reduces the problem to the investigation of periodic (with period 2π) solutions of a system of nonlinear equations whose right-hand sides are periodic functions of τ with the same period.

It is also not difficult to see that in the present case a small parameter is completely naturally introduced.

4. Self-synchronization of pendulum clocks suspended from a movable foundation (the problem of Huygens). As it has already been mentioned, the phenomenon of synchronization of dynamic systems was apparently first discovered experimentally by Ch. Huygens just in the case of the self-synchronization and phasing of the movements of two pendulum clocks suspended from a single thin beam. If we restrict the model of the clock motion to a single degree of freedom and if we assume that the clocks hang from an elastically suspended rigid platform having one degree of freedom (Fig. 5), then the equation of motion of the system coincides exactly with Equation (1.1) of the problem of self-synchronization of vibrators. In the clock problem the specific expression for the driving moments L_s and the resistive moments R_s will only be changed. Under the assumption stipulated above, these moments are to be considered dependent on the angle of rotation of the pendulums φ_s and the velocities $\varphi_s^{\cdot}$. The differential equations of motion of the clock-platform system can be written in the form

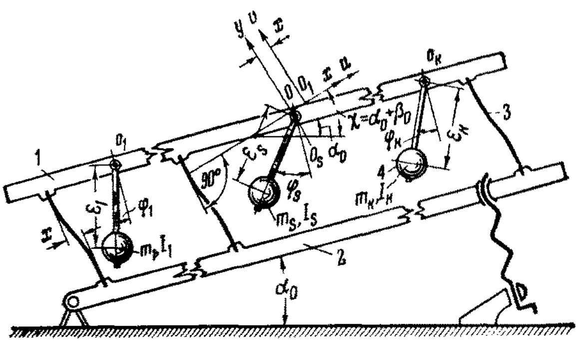

Fig. 5

$$\varphi_s^{\cdot\cdot} + \Omega_s^2\varphi_s = \frac{1}{I_s}\left[m_s\varepsilon_s x^{\cdot\cdot}\cos\chi + L_s(\varphi_s^{\cdot}, \varphi_s) - R_s(\varphi_s^{\cdot}, \varphi_s)\right]$$

$$Mx^{\cdot\cdot} + k_x x^{\cdot} + c_x x = \sum_{s=1}^{k} m_s\varepsilon_s\varphi_s^{\cdot\cdot}\cos\chi \qquad (s = 1, \ldots, k) \qquad (1.12)$$

where, in contrast to the problem of the vibrators, the angles of rotation of the pendulums are reckoned from the vertical and are considered small. By $\Omega_s = \sqrt{m_s g\varepsilon_s / I_s}$ we denote the frequencies of small free oscillations of the pendulums under the conditions that the points of support are immovable (the "partial frequencies" of oscillations).

The problem consists of the clarification of the conditions of existence and stability of periodic oscillations of the pendulums with common frequency ω, taking place in spite of a possible difference in the partial frequencies Ω_s of the separate pendulums, in the moments of inertia I_s, and in the moments L_s and R_s. In other words, we are concerned with the investigation of the conditions of existence and stability of periodic solutions of Equations (1.12) with a period $T = 2\pi/\omega$ which is unknown beforehand and is subject to determination in the process of solving the problem. We are also interested in computing (even though approximately) the actual periodic solutions.

Proceeding from the observations of Huygens, we may expect that synchronization of the pendulums will be possible only under conditions that the partial frequencies Ω_s differ slightly from one another. Taking into account this circumstance and considering also the possible orders of smallness of the separate quantities, it is natural to introduce into the system (1.12) a small parameter μ, which puts it into the form

$$\varphi_s^{\cdot\cdot} + \Omega_s^2\varphi_s = \mu\Phi_s^* (\varphi_s^{\cdot}, \varphi_s, x^{\cdot\cdot}) \qquad (s = 1, \ldots, k)$$
$$Mx^{\cdot\cdot} + c_x x = \sum_{s=1}^{k} m_s \varepsilon_s \varphi_s^{\cdot\cdot} \cos\chi - \mu k_x^* x^{\cdot} \tag{1.13}$$

where

$$\mu\Phi_s^* (\varphi_s^{\cdot}, \varphi_s, x^{\cdot\cdot}) = \Omega^2\chi_s\varphi_s + \frac{1}{I_s}[m_s\varepsilon_s x^{\cdot\cdot}\cos\chi + L_s(\varphi_s^{\cdot}, \varphi_s) - R_s(\varphi_s^{\cdot}, \varphi_s)]$$
$$k_x = \mu k_x^*, \qquad \Omega_s^2 = \Omega^2(1 - \chi_s) \tag{1.14}$$

and for Ω we may take either one of the Ω_s or some arbitrary average of the Ω_s.

We note that for $\mu = 0$ the first k equations become independent and the system (1.13) allows of a family of periodic solutions with period $2\pi/\omega$, depending on the $2k$ arbitrary constants. In this case, the characteristic equation of the generating system has at least the k-fold root $\rho = \exp(2\pi i\Omega)$ and k-fold root $\rho = \exp(-2\pi i\Omega)$. As above, as a consequence of the independence of the first k equations for $\mu = 0$, simple elementary devisors correspond to these roots.

5. S y n c h r o n i z a t i o n f o r t h e p a r a l l e l o p e r a t i o n o f e l e c t r i c a l m a c h i n e r y . We consider the formulation of the problem of the parallel operation of a certain number k of generators on a common load. The state of the s generator will be characterized by a single "rotational" coordinate, i.e. the angle of rotation of the rotor with respect to the stator φ_s, and likewise by a set of "oscillatory" state coordinates $x_1^{(s)}, \ldots, x_{r_s}^{(s)}$, which may be electrical as well as mechanical quantities.

First of all, let us assume that every generator operates on independent loads R_s, whose state is characterized by the state coordinates $u_1^{(s)}, \ldots, u_{v_s}^{(s)}$ (Fig. 6a). Then we shall have k independent autonomous systems in which under specific conditions there will be motion of the form

$$\varphi_s = \omega_s t + \psi_s(\omega_s t), \qquad x_j^{(s)} = x_j^{(s)}(\omega_s t), \qquad u_\rho^{(s)} = u_\rho^{(s)}(\omega_s t) \quad (1.15)$$
$$(j = 1, \ldots, r_s;\ \rho = 1, \ldots, \nu_s;\ s = 1, \ldots, k)$$

where ψ_s, $x_j^{(s)}$ and $u_\rho^{(s)}$ are periodic functions of time t with the period $2\pi/\omega_s$, and each of the ω_s is a constant which we may call the partial frequency of the generator corresponding to a given load R_s. Because of nonidentical loads, inaccuracy in manufacturing, and also because of imperfections in controls, the frequencies ω_s for the various generators, generally speaking, will be different.

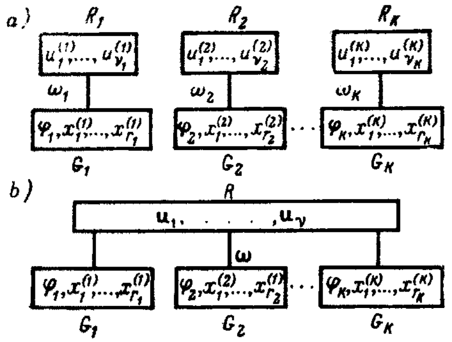

Fig. 6

We assume now that all of the generators are switched in parallel to operation on a single load R, the state of which is characterized by the state coordinates $u_1, \ldots, u_\nu$ (Fig.6*b*). Then the problem consists of finding conditions under which, notwithstanding the possible differences in partial frequencies ω_s, a rating with a common synchronous frequency ω is established in the integrated system. In other words, in the present case we are interested in clarifying the coditions for the existence and stability of motions of a combined system in the form

$$\varphi_s = \omega t + \psi_s(\omega t), \qquad x_j^{(s)} = x_j^{(s)}(\omega t), \qquad u_\rho = u_\rho(\omega t) \qquad (1.16)$$
$$(j = 1, \ldots, r_s;\ \rho = 1, \ldots, \nu;\ s = 1, \ldots, k)$$

where ψ_s, $x_j^{(s)}$ and u_ρ are periodic functions of t with the common period of $2\pi/\omega$, and where ω is a constant which is not known exactly beforehand. In other terms, the problem of synchronization arises here also*.

Here we do not give the differential equations of motion of the system being considered. In the general case they are so complicated that their derivation represents a nontrivial problem. In various cases of practical interest, these equations, written as a rule in terms of the functions ψ_s, $x_j^{(s)}$ and u_ρ, are given, for example in [16 to 18].

In terms of the variables indicated, the problem of the parallel operation of electrical synchronous machinery reduces, as in all of the problems examined above, to the study of the conditions for the existence and stability of periodic solutions of a certain system of differential equations whose right-hand sides are also periodic functions of time with the same period.

* We note that the term "synchronization" is often used in electrical engineering in a different sense than in the present paper.

6. Synchronization of vacuum-tube oscillators. The problem of the synchronization of vacuum-tube oscillators is of great significance in radio technology and in television. As an example we formulate the problem of the self-synchronization of the van der Pol type oscillators under the assumption that they are coupled together inductively. The differential equations of motion of the system will have the form

$$x_s^{\cdot\cdot} + \Omega_s^2 x_s = \mu\left[a_s(1 - x_s^2)\,x_s^{\cdot} + \sum_{j=1}^{k} b_{sj}x_j^{\cdot\cdot}\right] \qquad (s = 1, \ldots, k) \qquad (1.17)$$

Here $\Omega_s > 0$, $\mu > 0$ and $a_s > 0$ are constants. For $b_{sj} = 0$ Equations (1.17) pass into k independent nonlinear equations, known under the name of the equations of van der Pol. Such equations have a periodic solution of period $T_s = 2\pi\,[1 + \delta(\mu)]\,/\,\Omega_s$, where $\delta(0) = 0$; therefore in the case of absence of coupling between oscillators ($b_{sj} = 0$) each of them generates oscillations in the steady state operational condition whose frequencies are $\omega_s(\mu) = \Omega_s\,/\,[1 + \delta_s(\mu)]$, which, generally speaking, are different. These frequencies may be called the partial frequencies of the oscillators.

As before, the basic problem will be the establishment of conditions under which all of the coupled oscillators operate with a common frequency ω (unknown beforehand), notwithstanding the possible differences in the partial frequencies $\omega_s(\mu)$. In other words, the matter again concerns the finding of conditions of the existence and stability of periodic solutions of the system (1.17).

If the frequencies $\Omega_s = \omega_s(0)$ differ slightly from one another so that one may set $\Omega_s^2 = \Omega^2(1 - \mu\chi_s)$, then equations (1.17) reduce to the form

$$x_s^{\cdot\cdot} + \Omega^2 x_s = \mu\Phi_s(x_s^{\cdot}, x_s; x_1^{\cdot\cdot}, \ldots, x_k^{\cdot\cdot}) \qquad (s = 1, \ldots, k) \qquad (1.18)$$

where

$$\mu\Phi_s(x_s^{\cdot}, x_s; x_1^{\cdot\cdot}, \ldots, x_k^{\cdot\cdot}) = \mu\left[a_s(1 - x_s^2)\,x_s^{\cdot} + \chi_s\Omega^2 x_s + \sum_{j=1}^{k} b_{sj}x_j^{\cdot\cdot}\right] \qquad (1.19)$$

The equations (1.18) are a quasilinear autonomous system which is analogous to the first kth equations (1.13) of the Huygens problem. As before, the generating system for (1.18) has a periodic solution of period $2\pi/\Omega$, depending on the $2k$ arbitrary constants, whereas the characteristic equation of this system has two k-fold roots $\rho = \exp(\pm 2\pi_i\Omega)$ with simple elementary devisors.

2. The general formulation of the problem of synchronization. In the general form, the problem of the synchronization of dynamic systems may be formulated in the following manner.

We consider a certain number k of dynamic objects assembled in one single system (Fig. 7). Let the motion of the sth object be defined by a r_s-dimensional vector $\mathbf{x}^{(s)} = [x_1^{(s)}, \ldots, x_{r_s}^{(s)}]$ $(s = 1, \ldots, k)$, the components of which $x_j^{(s)}$ will be coordinates of the object in the state space of the system.

The motion of the system as a whole will be specified by the aggregate of the vectors $\mathbf{x}^{(s)}$ introduced above and by the ν-dimensional vector $\mathbf{u} = [u_1, \ldots, u_\nu]$ which characterizes the connections between the objects. Hence, the state space of the system has $l = r_1 + \ldots + r_k + \nu$ dimensions.

Let the motion of the system under consideration be described by differential Equations

$$\mathbf{x}^{\cdot(s)} = \mathbf{X}^{(s)}(\mathbf{x}^{(s)}) + \mathbf{F}^{(s)}(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}, \mathbf{u}) \qquad (s = 1, \ldots, k) \qquad (2.1)$$
$$\mathbf{u}^{\cdot} = \mathbf{U}(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}, \mathbf{u})$$

where

$$\mathbf{X}^{(s)} = [X_1^{(s)}, \ldots, X_{r_s}^{(s)}],\ \mathbf{F}^{(s)} = [F_1^{(s)}, \ldots, F_{r_s}^{(s)}],\ \mathbf{U} = [U_1, \ldots, U_\nu]$$

are, respectively, r_s and ν-dimensional vector functions, satisfying extremely general requirements under which the system (2.1) will be dynamical, and certain special requirements which will be indicated below. The vector-functions $\mathbf{F}_s$ and $\mathbf{U}$, characterizing the connections between the specific objects will be called the constraint functions.

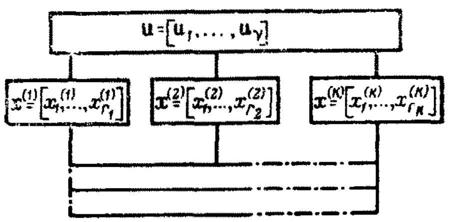

Fig. 7

From the block diagram given in Fig. 7, and likewise from an examination of Equations (2.1), it is seen that each of the objects may be connected to the remaining objects both directly and through a system of constraints, the state of which is characterized by the state coordinates u_ρ. Further, it is seen from the equations that the coordinates $x_j^{(s)}$ which specify the state of objects and the coordinates u_ρ of the system of constraints enter, in essence, into Equations (2.1) on a completely "equal footing". The specific character of each group of variables in many problems of synchronization will be clarified below.

We shall consider the basic problem of the theory of synchronization to be the establishment of the conditions for the existence and stability of solutions of Equations (2.1) in the form

$$x_j^{(s)} = \sigma_j^{(s)}[q_j^{(s)}\omega t + y_j^{(s)}(\omega t)] \qquad (j = 1, \ldots, r_s;\ s = 1, \ldots, k) \qquad (2.2)$$
$$u_\rho = \sigma_\rho[q_\rho \omega t + v_\rho(\omega t)] \qquad (\rho = 1, \ldots, \nu)$$

where ω is a positive constant, $y_j^{(s)}(\omega t)$ and $v_\rho(\omega t)$ are periodic functions of ωt with the period 2π; $q_j^{(s)}$ and q_ρ are numbers each of which may be either zero or unity. In the first case we shall conditionally call the corresponding coordinates $x_j^{(s)}$ or u_ρ oscillatory, in the second case we shall call them rotational. By $\sigma_j^{(s)}$ and σ_ρ we shall denote numbers which may arbitrarily be either $+1$ or -1.

To solutions of the form (2.2) there correspond either oscillatory motions or motions that are uniform on the average in which each coordinate has the same frequency (linear or angular velocity) ω. In the sequel we shall call such motions synchronous.

We shall now concretely specify the right-hand sides of Equations (2.1). We shall assume that the functions $X_j^{(s)}$, $F_j^{(s)}$, U_ρ depend on their arguments in such a way that upon substitution for $x_j^{(s)}$ and u_ρ in accordance with Formulas (2.2), these functions will become periodic functions of the "nondimensional time" $\tau = \omega t$ with period 2π.

The last condition is not necessary for the existence of synchronous motion in system (2.1), however, it is fulfilled in all concrete problems of synchronization known to us and it essentially simplifies their solution. Hence, in what follows, this condition will be assumed to be satisfied. We note that for this condition to hold it is sufficient that $X_j^{(s)}$ and U_ρ be periodic functions of the rotational coordinates with period 2π, and likewise, that they be functions perhaps, of the differences $\sigma_j^{(s)}x_j^{(s)} - \sigma_n^{(m)}x_n^{(m)}$, $\sigma_j u_j - \sigma_\rho u_\rho$, $\sigma_j^{(s)}x_j^{(s)} - \sigma_\rho u_\rho$, where $x_j^{(s)}$, $x_n^{(m)}$, u_j and u_ρ are rotational coordinates.

Using Formulas (2.2) and passing in Equations (2.1) from the variables **x**, **u** and t, to the variables **y**, **v** and $\tau = \omega t$, we obtain a system of the form

$$\mathbf{y}^{(s)\prime} = \mathbf{Y}^{(s)}(\mathbf{y}^{(s)}, \tau) + \mathbf{\Phi}^{(s)}(\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(k)}, \mathbf{v}, \tau) \qquad (s = 1, \ldots, k) \tag{2.3}$$
$$\mathbf{v}' = \mathbf{V}(\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(k)}, \mathbf{v}, \tau)$$

where in correspondence with the assumptions made on the character of the vector-functions $\mathbf{X}^{(s)}, \mathbf{F}^{(s)}$ and $\mathbf{U}$, the functions $\mathbf{Y}^{(s)}$, $\mathbf{\Phi}^{(s)}$ and $\mathbf{V}$ will be periodic with respect to the nondimensional variable τ with period 2π.

Thus the basic problem of synchronization reduces to establishing conditions on the existence and stability of periodic solutions of the system of equations (2.3) with period 2π.

In addition to the problem of synchronization that has been formulated above, the following problems are also often of interest.

1. The actual calculation of the synchronous angle or linear velocity (frequency) ω, and likewise the solutions of (2.2) which correspond to the stable synchronous motions. In many cases one may restrict himself to the determination of average values over a period 2π of the functions $\mathbf{y}^{(s)}(\omega t)$ and $\mathbf{v}(\omega t)$, that is the quantities

$$\boldsymbol{\alpha}^{(s)} = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{y}^{(s)}(\tau)\, d\tau, \qquad \boldsymbol{\alpha} = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{v}(\tau)\, d\tau \tag{2.4}$$

and likewise to the maximum deviations $|y_j^{(s)}(\tau) - \alpha_j^{(s)}|_{\max}$ and $|v_\rho(\tau) - \alpha_\rho|_{\max}$ from these average values.

2. The choice of a system of constraints under which the existence and stability of synchronous motions of (2.2) of the given form is guaranteed. This problem, which may be called the problem of synthesis, is in a certain sense the inverse of the basic problem.

Also, sometimes of interest is the extremely difficult problem of determining a region of initial values (the "capture region") in the phase space of the system such that for subsequent times the motion will approach the specified synchronous motion without restriction.

What was indicated above was related to the problem of simple synchronization. However the more complicated problem of multiple synchronization also arises in a number of applications. The question then is not rotation or oscillation with a common angular velocity (frequency), but with common velocities (frequencies) of the type $n_j\omega$, where n_j is an integer, generally speaking, different for the various components of the vectors $\mathbf{x}^{(s)}$ and $\mathbf{u}$.

One of the most important classes of problems of synchronization is formed by problems of synchronization of self-oscillating objects. As a rule these objects are of the same type and each is isolated from the rest (the constraint functions $\mathbf{F}^{(s)}$ and $\mathbf{\Phi}^{(s)}$ in Equations (2.1) and (2.3) are absent) and under specific conditions may perform motions of the type (2.2) which are characterized by a certain frequency (angular or linear velocity) ω_s. It is natural to call the quantity ω_s the partial frequency (velocity) of the object. The problem of synchronization consists of finding the conditions under which all the objects upon assembling into a single system may perform motion of the same type but with an identical frequency (velocity) ω or likewise with frequencies (velocities) of the form $n_s\omega$.

Depending on the character of the formulation of the problem of synchronization of self-oscillating objects or systems containing such objects, it is necessary to distinguish between the problem of internal (autonomous) synchronization and the problem of external (nonautonomous) synchronization.

In the first, more general case, to which the above formulated problem of synchronization is related, all of the objects to be synchronized are considered as elements on an equal footing of a single autonomous dynamical system. In this case, the frequency of synchronous motion is established as a result of the interaction of all of the elements of the system. The right-hand sides of equations (2.1) in this situation do not contain the time t in explicit form, and the value of the synchronous frequency ω is not known beforehand and is subject to determination in the process of solving the problem (see Subsections 1 and 3 to 6, Section 1).

In the second case it is assumed that one of the self-oscillating objects that is to be synchronized is significantly stronger than the remaining objects and therefore its motion is to be considered independent of the character of the motions of the remaining elements of the system. The singled-out object acts on the other elements of the system and therefore the frequency (or angular velocity) of the synchronous motion is assumed to be given at the outset and is unchanging.

In such approach to a problem, the initial system (2.1) becomes nonautonomous and thereby its order is lowered (see Subsection 2 in Section 1).

It is not hard to see that all of the concrete problems that were considered in Section 1 are particular cases of the general problem that has been formulated here.

In conclusion we note that in a number of applications the study of the synchronization of a system with distributed parameters is of interest. In this case, a number of equations (2.1) are partial differential equations. Clearly, all that has been indicated above may be also extended to such a system of equations.

3. Basic peculiarities of the problems of synchronization. The differential equations of the problems of synchronization, as a rule, are essentially nonlinear. However it is often possible to introduce into them a small parameter. This allows one to apply methods from the theory of periodic solutions which was developed by A. Poincaré and A.M. Liapunov.

One can point out two important groups of problems of synchronization in which the method of a small parameter can be effectively applied.

In the first group are problems of synchronization involving objects which are "weakly" coupled. These are the very problems of interest in application. On the one hand, synchronization is technically the most straightforward and most economical by means of "weak" interconnections. On the other hand, if it is necessary to apply "strong" interconnections between some of the objects then, as a rule, the system may, after the application of the interconnections, be regarded as a single system for which the problem of synchronization does not arise. Thus, for example, two mechanically unbalanced vibrators,whose shfts are connected by means of drive gears with rigid intermediate elements that are between the vibrators and the gears, form, practically speaking, a single two-shafted vibrator.

Most of the concrete problems of synchronization examined in Section 1 may be put into the category of synchronization with weak coupling, as well as many other problems.

In the case of objects with weak coupling, the basic equations (2.1) may be represented in the form

$$\mathbf{x}^{\cdot(s)} = \mathbf{X}^{(s)}(\mathbf{x}^{(s)}) + \mu\mathbf{F}^{\circ(s)}(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}, \mathbf{u}, \mu) \qquad (s = 1, \ldots, k)$$
$$\mathbf{u}^{\cdot} = \mathbf{U}^{*}(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}, \mathbf{u}, \mu) \qquad (3.1)$$

or in terms of the variables $\mathbf{y}^{(s)}$, $\mathbf{v}$ and $\tau = \omega t$ into the form

$$\mathbf{y}'^{(s)} = \mathbf{Y}^{(s)}(\mathbf{y}^{(s)}, \tau) + \mu\mathbf{\Phi}^{\circ(s)}(\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(k)}, \mathbf{v}, \tau, \mu) \qquad (s = 1, \ldots, k)$$
$$\mathbf{v}' = \mathbf{V}^{*}(\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(k)}, \mathbf{v}, \tau, \mu) \qquad (3.2)$$

Here $\mathbf{F}^{\circ(s)}$, $\mathbf{U}^{*}$, $\mathbf{\Phi}^{\circ(s)}$ and $\mathbf{V}^{*}$ are vector-functions of the same class as $\mathbf{F}^{(s)}$, $\mathbf{U}$, $\mathbf{\Phi}^{(s)}$ and $\mathbf{V}$ in Equations (2.1) and (2.3), whereby $\mathbf{F}^{\circ(s)}$, $\mathbf{U}^{*}$, $\mathbf{\Phi}^{\circ(s)}$ and $\mathbf{V}^{*}$ are likewise functions of a small parameter μ which it is sufficient to consider an analytic for μ in the interval $|\mu| < \mu_0$, where $\mu_0 > 0$.

An extremely wide class of problems is formed by problems of synchronization of identical or almost identical objects which are weakly coupled to one another. In this case, the functions $\mathbf{X}^{(s)}$ and $\mathbf{Y}^{(s)}$ in Equations (3.1) and (3.2) do not depend on the index s.

Equations (3.2) which have been introduced for the study of periodic solutions and whose solution in the present case gives the solution of the basic problem of synchronization, have the property that in the corresponding generating system each of the first k equations are independent. After the vectors $\mathbf{y}_0^{(s)}$ have been determined from them, the vector $\mathbf{v}_0$ may be found from the last equation. This circumstance essentially simplifies the solution of the generating system; however, it leads to a number of complications in the study of the complete system. The fact is that the generating system in synchronization problems allows not of a single solution but of a whole family of periodic solutions

$$y_{j0}^{(s)} = y_{j0}^{(s)}(\tau, \alpha_1, \ldots, \alpha_p) \qquad (j = 1, \ldots, r_s;\ s = 1, \ldots, k) \qquad (3.3)$$

which depend on a certain number p of arbitrary parameters α_q. In this case, p is equal to or greater than the number of objects k, which is not hard to see. In the generating approximation the equations of motion of the objects are independent, and if they have a periodic solution $y_{j0}(\tau)$, then, in accordance with the autonomous nature of each of Equations (3.1) and in accordance with (2.2), they also allow of the periodic solutions $y_{j0}^{(s)}(\tau + \omega\alpha_s) + q_j^{(s)}\omega\alpha_s$, where α_s are arbitrary constants.

It is known that the presence of solutions of the type (3.3) in the generating system corresponds to a singular case wherein the Poincaré determinant goes to zero together with its minors up to order $l - p + 1$, inclusively [19]. In this case not only the study of the existence, but of the stability of periodic solutions, becomes more complicated since, by virtue of the theorem of Poincaré, the characteristic equation for the system corresponding to the generating system and generating solution always has a p-fold root which is equal to unity. Therefore, the original approximation to the roots of the characteristic equation does not answer the question of stability and it is necessary to study higher order approximations.

The presence of multiple (or near by) roots in the characteristic equation of the synchronization problem may be associated not only with autonomy but with the presence in the system of a number of identical (or almost identical) objects. We note also that in systems with weak coupling, the generating equations of motion of all of the objects are independent. Hence, the characteristic equation will disintegrate into not less than k independent equations and the elementary devisors corresponding to each root will be simple only if each of the equations (3.2) separately has simple roots.

There also exists another category of synchronization problems in which the method of small parameters may be effectively used. These are cases in which the study of synchronization may be restricted to conditions under which the object performs motions that are close to some known motions.

So, for example, a number of problems in the synchronization of vibrators, self-balancing machines, bending-torsional oscillations of shafts and the synchronization of generators may often be sufficiently solved under the assumption that in synchronous motion all of the rotating coordinates (angles of rotation of the rotors) change with time in a neighborhood of an equilibrium rotation with the synchronous angular velocity ω .

4. Short review of works on the theory of synchronization of dynamic systems. On some unsolved problems.

1. Mathematical investigations. A systematic study of the general case when the generating solutions depend on a number p of arbitrary parameters α_j was begun in the monograph of Malkin [19]. As was indicated in Section 3, this case is of particular interest in the theory of synchronization. Generalizing the results of Poincare [20], Malkin established that the periodic solutions of the basic system of equations which transform into the generating solution for $\mu = 0$ may correspond only to those values of the parameters α_j which satisfy a certain system of equations

$$P_s(\alpha_1, \ldots, \alpha_p) = 0 \qquad (s = 1, \ldots, p) \tag{4.1}$$

To each simple solution $\alpha_1 = \alpha_1^*, \ldots, \alpha_p = \alpha_p^*$ of this system there indeed uniquely corresponds a periodic solution of the basic system of differential equations which is analytic in μ and which for $\mu = 0$ goes into the generating solution.

Later, extensions were obtained or concrete methods of constructing the functions P_s were indicated for various types of systems of differential equations by Malkin [21 and 22], Shimanov [23 to 28], Merman [29], Coddington and Levinson [30], Bulgakov [31], Volk [32], Neimark [33], Volosov [34], Neimark and Shil'nikov [35], Kolovskii [36], Kushul [37] and Rodinov [38]. Likewise, in the work of Merman [29], and Shimanov [23 and 26] and certain other authors particular cases were studied in which the solution of equations (4.1) is not simple. For systems with one or two degrees of freedom, for $p = 2$ or $p = 4$, these cases were studied in detail by Proskuriakov [39 and 40], and Plotnikova [41 and 42].

The problem studied by the author in [3] on the synchronization of vibrators was an example of a concrete technological problem the solution of which immediately required a study of a case wherein the generating solution depended on p arbitrary parameters and the characteristic equation for the generating system and the generating solution had a p-fold root equal to unity (with simple elementary devisors). In this paper it was shown that the study of the stability of periodic solutions for sufficiently small μ may be reduced to the study of the signs of the real parts of the roots of an algebraic equation of the pth degree (δ_{sj} is the Kroncker symbol).

$$\left|\frac{\partial P_s}{\partial \alpha_j} - \delta_{sj}\varkappa\right|_{\alpha=\alpha^*} = 0 \quad (s,\ j = 1,\ldots,p) \tag{4.2}$$

that is to the usual problem of Hurwitz.

In [43] a corresponding theorem on stability was proved for quasilinear nonautonomous systems. A proof of a number of analogous theorems for periodic solutions of nonautonomous nonlinear systems was given in the monograph of Malkin [44]. The case of quasilinear autonomous systems was examined in our note [63], and a generalization to almost-periodic oscillations of quasilinear systems with delay was given by Shimanov [45]. Nohel [46] obtained certain of the results of our work [43 and 63] by means of some other arguments and also obtained a number of new theorems.

In [47] it was noted that if there exists a function $D(\alpha_1, \ldots, \alpha_p)$, such that $\partial D / \partial \alpha_j = -P_j(\alpha_1, \ldots, \alpha_p)$, then this function plays the same role in the problem of periodic solutions that the potential energy plays in the problem of equilibrium positions. Stationary points of the function D may correspond to periodic solutions of the investigated type, whereas minimum points, obtained from an analysis of second order terms in the expansion of D, may correspond to stable periodic motions. It was established in a number of cases that the function D coincides in an average over a period with the value of kinetic potential of the system. This circumstance was noted for the first time in a particular case in a paper by Lavrov and the author [48].

In the paper by Bakhmutskii [49] it was shown that by modifying some of the arguments, the method of Poincaré may be successfully applied to the study of the processes leading to the establishment of periodic solutions. Usually this is done by means of asymptotic methods [50]. In this connection, exactly the case of the presence of excited solutions of the type (3.3) was studied in [49] and it was established that in the initial approximation the parameters α_j may, generally speaking, be assumed to be slowly-varying functions of time which are determined from the system of the equations

$$\frac{d\alpha_s}{d\tau} = \frac{\mu}{2\pi} P_s(\alpha_1, \ldots, \alpha_p) \qquad (s = 1, \ldots, p) \tag{4.3}$$

2. Study of individual classes of dynamical systems by means of the theory of synchronization. The synchronization of weakly-coupled self-oscillating objects with almost uniform rotating motion was examined in [47] and in more detail in the author's dissertation, submitted in 1962 to the M.I. Kalinin Polytechnic Institute in Leningrad. It was assumed that the motion of the system is described by Equations

$$\begin{aligned} &I_s\varphi_s^{\cdot\cdot} + k_s\varphi_s^{\cdot} - k_s\sigma_s\omega = \mu\Phi_s && (s = 1, \ldots, k) \\ &\frac{d}{dt}\frac{\partial L}{\partial x_r^{\cdot}} - \frac{\partial L}{\partial x_r} = Q_r^{\circ} + \mu Q_r^{(1)} && (r = 1, \ldots, \nu) \end{aligned} \tag{4.4}$$

where

$$\mu\Phi_s = I_s\varphi_s^{\cdot\cdot} + k_s\varphi_s^{\cdot} - k_s\sigma_s\omega - \frac{d}{dt}\frac{\partial L}{\partial \varphi_s^{\cdot}} + \frac{\partial L}{\partial \varphi_s} + Q_s$$

and $L = T - \Pi$ is the Lagrangian, φ_s is a rotational generalized coordinate, x_r is an oscillatory generalized coordinate, Q_s and $Q_r^{\circ} + \mu Q_r^{(1)} = Q_r$ are generalized forces, I_s, k_s and ω are positive constants, and $\sigma_s = \pm 1$. It was also assumed that the generating system corresponding to Equations (4.4), had solutions of the type

$$\varphi_s = \sigma_s(\omega t + \alpha_s) \qquad (s = 1, \ldots, k)$$
$$x_r{}^\circ = x_r{}^\circ(\omega t, \alpha_1, \ldots, \alpha_k) \qquad (r = 1, \ldots, \nu) \tag{4.5}$$

where x_r° are periodic functions of time t with period $2\pi/\omega$ and α_s are constants.

This class of problem includes problems of synchronization of mechanical vibrators, automatic balancing, bending-torsional oscillations of shafts with disks, and likewise many problems of synchronization of electrical machinery (see Section 1, Subsections 1 to 3 and 5).

Under sufficiently general assumptions on the form of the functions L and Q it was shown that the functions P_s, on which, in accordance with the above, depends the solution of the problem of existence and stability of synchronous motion, may be represented in the form

$$P_s(\alpha_1, \ldots, \alpha_k) = \frac{1}{k_s}\left\{\frac{\partial\Lambda}{\partial\alpha_s} + \frac{\omega}{2\pi}\sum_{r=1}^{\nu}\int_0^{2\pi/\omega}[Q_r{}^\circ]\frac{\partial x_r{}^\circ}{\partial\alpha_s}dt + \frac{\omega\sigma_s}{2\pi}\int_0^{2\pi/\omega}[Q_s]\,dt\right\}$$
$$(s = 1, \ldots, k) \tag{4.6}$$

where

$$\Lambda = \Lambda(\alpha_1, \ldots, \alpha_k) = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}[L]\,dt$$

and the square brackets indicate that the quantities included in them are to be for the generating solution.

If, as is often the case, $Q_r^\circ = 0$ and

$$L = L_0 + L_1, \qquad L_0 = \frac{1}{2}\sum_{r=1}^{\nu}\sum_{j=1}^{\nu}(a_{rj}x_r^{\cdot}x_j^{\cdot} - b_{rj}x_r x_j)$$
$$L_1 = \frac{1}{2}\sum_{s=1}^{k}\sum_{j=1}^{k}d_{sj}\varphi_s^{\cdot}\varphi_j^{\cdot} + \sum_{r=1}^{\nu}f_r x_r^{\cdot} + \sum_{s=1}^{k}F_s \tag{4.7}$$

where a_{rj}, b_{rj} and d_{sj} are constants, f_r are functions of φ_j and $\varphi_j^{\cdot}$, and F_s are periodic functions of φ_s with period 2π, then one may apply [47]

$$\Lambda = -\Lambda_0 = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}[L_0]\,dt \tag{4.8}$$

Finally, if the last two terms in Equations (4.6) can be represented as the derivatives with respect to α_s of some function A, and if all of the k_s are identical, then there exists a function D which was discussed in the previous subsection.

In the above mentioned dissertation, the author also studied the problem of synchronization of weakly coupled van der Pol oscillators. In the particular case of two oscillators this problem was studied earlier by Minorski by other methods [51].

The problem of internal synchronization of almost identical autonomous objects under linear weak coupling was examined in a paper by Nagaev [52].

3. P a p e r s o n t h e t h e o r y o f s y n c h r o n i z a t i o n o f s p e c i f i c m a c h i n e s . As far as concrete problems of synchronization are concerned, those considered in greatest detail are problems of the simultaneous parallel operation of a number of synchronous electrical machines (see footnote on page 249). Among the first investigations in this field one should mention the work of Ollendorf and Peters [53], Krylov and Bogoliubov [16], Zhdanov and Lebedev [17], and Gorev [18]. A detailed bibliography and a description of the present state of this problem may be found in [54 to 56].

Because of its extreme complexity, many important aspects of the problem remain unstudied until the present day, notwithstanding the presence of numerous investigations in which a number of substantial results have been obtained. In particular, almost unexamined is the nonsymmetrical range of operation of machines in the general case in which transient phenomena are described by equations with periodic coefficients.

The self-synchronization of two coupled relaxation oscillators (multivibrators) was studied by Bremzen and Fainberg [57]. They discovered a range of multiple synchronization and showed the possibility of oscillations in the coupled system in a case in which neither of the oscillators was excited.

Associated with the problem of synchronization of vacuum-tube oscillators is the related problem of the self-oscillations of coupled circuits, one of which is self-excited and the other of which is nonexcited. This problem was already posed by van der Pol. The first to examine it in a sufficiently rigorous formulation for the case of strong coupling between the circuits were Andronov and Vitt [58], and Skibarko and Strelkov [59].

A study of corresponding case for weak coupling was carried out by Belliustin [60]. A mechanical analogue of the system indicated above, but with impulse excitation, was studied in a paper by Butenin [61] which was devoted to the solution of a problem by Kelvin from the theory of clocks. A system of a number of strongly coupled RLC circuits, one of which is excited, was examined by Gushchin [62] in connection with the theory of a dynamic flip-flop.

The Huygens problem of the self-synchronization of pendulum clocks was studied by Minorski in the particular case of two clocks by means of the so-called stroboscopic method [51]. The corresponding problem but for an arbitrary number of clocks and in a more rigorous formulation (see Section 1, Subsection 4), was studied by us in the dissertation mentioned above by means of similarity theorems [63].

Another group of papers consists of investigations of the theory of synchronization of mechanical vibrators. Paper [3] explained the phenomenon of self-synchronization of vibrators and studied the simplest case when the operating element of the machine has one degree of freedom in all (see Section 1, Subsection 1). The more complicated problem of the self-synchronization of vibrators fixed to machines which contain a vibrating element and which may perform arbitrary planar motion was studied by the author in [4 and 5]. The foundations of the theory of forced electrical synchronization, and also synchronization by means of the introduction of elastic elements between the rotors of the vibrators, was studied in [11 and 64]. Investigations [48 and 65] were devoted to an integral test for stability of motion in problems of self-synchronization of vibrators. A further generalization of this test was given in [47]. In the paper of Shekhter [66] the problem of self-synchronization of vibrators in machines with a two degree of freedom vibrating element was studied in connection with the installation of vibrationally sunk shells. Applications of the theory of synchronization of vibrators to the dynamics of crushing and grinding machines, transport apparatus, and to certain other vibrating machinery was examined in [5, 7, 67 and 68], and likewise in the indicated dissertation of the author.

In the paper by Lavrov [70], the problem of the synchronization of vibrators fixed to a free rigid body was studied. In this connection, a case was examined in which among the vibrators there were so-called rocking vibrators.

In the monograph [71] Ragul'skis studied a number of s stems with self-synchronized vibrators, among them the simplest case of multiple self-synchronized vibrators and self-synchronization in the presence of shock. A complicated system with shocks in which oscillations are excited by two self-synchronized vibrators was studied in connection with the theory of vibrating jaw crushers by Nagaev [72].

Problems of automatic balancing and of bending-torsional oscillations of a rotating shaft with unbalanced disks which, in accordance with the presentation in Section 1, may be considered as problems of synchronization were studied by the author in the above-mentioned dissertation. In another formulation and by means of other methods the first of these problems was studied earlier by Detinko [15].

A degenerate class of problems in the theory of synchronization are so-called capture problems in which essentially externally excited synchronization of a single unique self-oscillating object is involved. Work on the theory of capture due to van der Pol [73], Appleton [74], and Andronov and Vitt [58] and their numerous successors played an important role in the development of the general theory of nonlinear oscillations.

In the category of capture phenomenon one may place the peculiar effect of excitation and sustenance of rotation of an unbalanced rotor by means of oscillation of its axis. This effect was studied by Bogoliubov [75], afterwards by the author [6] and later by Barkan and Shekhter [76], Gortinskii [77], Caughey [78], and Ragul'skis [71]. The connection of this effect with the phenomenon of self-synchronization was established by us in [4].

Among problems in the theory of synchronization which have not been completely resolved up to the present time, we mention problems of synchronization in systems with discontinuous characteristics (in particular, the problem of synchronization of vibrators in systems with impulses), problems of multiple synchronization, the problem of synchronization for system with distributed parameters, question of sunchronization (self-organization) of biological and other objects, generally speaking, of nondynamical character, problems of finding "regions of attraction" of synchronous motions in the phase space of the system, and questions of the validity of the transition from the study of problems of autonomous (internal) synchronization to problems of nonautonomous (external) synchronization (see Section 2).

In conclusion we mention the following. In the overwhelming majority of problems studied on internal synchronization of weakly-coupled self-oscillating objects, the corresponding system of differential equations, as a rule, allowed of at least one stable periodic solution (that is, synchronization took place), only if there were sufficiently small differences between the partial frequencies or the angular velocities. This confirms the fact that the tendency toward synchronization is a general regularity of behavior of interconnected material objects.

In addition, synchronization sometimes takes place, notwithstanding the weakness of the connections, even in the face of the existence of differences in partial frequencies and in other parameters of the separate objects [4 and 5].

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